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Note

On the absolute (C, α) convergence for functions of bounded variation ☆

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Let f be a 2π-periodic function integrable on $[-\pi, \pi]$. For $\alpha > -1$, the Cesàro means of order α of the Fourier series of f are defined by

$$\sigma_n^\alpha(f)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_n^\alpha(t)\,dt,$$

where

$$K_n^\alpha(t) = \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}D_k(t), \quad A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}, \quad D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}.$$

Define

$$R_n^\alpha(f, x) = \sum_{k=n+1}^{\infty} |\sigma_k^\alpha(f)(x) - \sigma_{k-1}^\alpha(f)(x)|.$$

The following result was presented in [1]:

Theorem HB. *Let* $x \in [0, \pi]$ *and* f *a* 2π*-periodic function of bounded variation on* $[-\pi, \pi]$. *Then, for* $\alpha > 0$ *and* $n \geqslant 2$, *we have*

$$R_n^\alpha(f, x) \leqslant \frac{4\alpha}{n\pi}\sum_{k=1}^{n} V_0^{\frac{\pi}{k}}(\varphi_x),$$

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where

$$\varphi_x(t) := f(x+t) + f(x-t) - f(x+0) - f(x-0)$$

and $V_a^b(f)$ *denotes the total variation of* f *on* $[a,b]$.

But this is incorrect when $0<\alpha<1$. Our result is the following theorem.

Theorem. *Suppose* $0<\alpha<1$, $x\in[0,\pi]$ *and* f *is a* 2π*-periodic function of bounded variation on* $[-\pi,\pi]$. *Then for* $n\geqslant 2$

$$R_n^\alpha(f,x) \leqslant \frac{100}{\alpha^2 n^\alpha} \sum_{k=1}^{n} k^{\alpha-1} V_0^{\frac{\pi}{k}}(\varphi_x),$$

and there exists a 2π*-periodic function* f^* *of bounded variation on* $[-\pi,\pi]$ *and a point* $x\in[0,\pi]$ *such that*

$$R_n^\alpha(f^*,x) > \frac{1}{20000\alpha n^\alpha} \sum_{k=1}^{n} k^{\alpha-1} V_0^{\frac{\pi}{k}}(\varphi_x) \quad (n\geqslant 8).$$

We need two lemmas.

Lemma 1. *Let* $0<\alpha<1$. *Define*

$$\gamma_n^\alpha(t) = K_n^\alpha(t) - K_{n-1}^\alpha(t) - \frac{\cos[(n+\frac{\alpha}{2})t - \frac{\pi}{2}\alpha]}{A_n^\alpha (2\sin\frac{t}{2})^\alpha} \quad 0<t\leqslant\pi.$$

Then for $\frac{\pi}{n}\leqslant t\leqslant\pi$, $n\geqslant 2$

$$|\gamma_n^\alpha(t)| \leqslant \frac{16\alpha}{(nt)^{\alpha+1}}. \tag{1}$$

Proof. It follows from formula (5.15) of [2, p. 95] that

$$K_n^\alpha(t) = D_n^\alpha(t) + E_n^\alpha(t) \quad (0<t\leqslant\pi),$$

where

$$D_n^\alpha(t) := \frac{1}{A_n^\alpha} \frac{\sin[(n+\frac{1+\alpha}{2})t - \frac{\pi}{2}\alpha]}{(2\sin\frac{t}{2})^{\alpha+1}},$$

$$E_n^\alpha(t) := -\Re\left\{ \frac{e^{i(n+\frac{1}{2})t}}{A_n^\alpha (2\sin\frac{t}{2})} \sum_{v=n+1}^{\infty} A_{v+1}^{\alpha-2} \frac{e^{-i(v+1)t}}{1-e^{-it}} \right\} + \frac{\alpha}{n+1} \frac{1}{(2\sin\frac{t}{2})^2}.$$

Since

$$D_n^\alpha(t) - D_{n-1}^\alpha(t) = \frac{\cos[(n+\frac{\alpha}{2})t - \frac{\pi\alpha}{2}]}{A_n^\alpha (2\sin\frac{t}{2})^\alpha} - \frac{\alpha\sin[(n+\frac{\alpha-1}{2})t - \frac{\pi\alpha}{2}]}{nA_n^\alpha (2\sin\frac{t}{2})^{\alpha+1}},$$

we have

$$\gamma_n^\alpha(t) = -\frac{\alpha \sin[(n+\frac{\alpha-1}{2})t - \frac{\pi\alpha}{2}]}{nA_n^\alpha (2\sin\frac{t}{2})^{\alpha+1}} + E_n^\alpha(t) - E_{n-1}^\alpha(t).$$

Hence

$$|\gamma_n^\alpha(t)| \leqslant \frac{\alpha}{nA_n^\alpha (2\sin\frac{t}{2})^{\alpha+1}} + |E_n^\alpha(t) - E_{n-1}^\alpha(t)|. \tag{2}$$

A straightforward calculation gives

$$E_n^\alpha(t) - E_{n-1}^\alpha(t) = \Re\left\{\frac{e^{-\frac{it}{2}}}{2\sin\frac{t}{2}(1-e^{-it})}\sum_{v=1}^{\infty}\left(\frac{A_{v+n}^{\alpha-2}}{A_{n-1}^\alpha} - \frac{A_{v+1+n}^{\alpha-2}}{A_n^\alpha}\right)e^{-ivt}\right\} - \frac{\alpha}{n(n+1)}\frac{1}{(2\sin\frac{t}{2})^2}.$$

Then we obtain

$$\begin{aligned}|E_n^\alpha(t) - E_{n-1}^\alpha(t)| &\leqslant \frac{1}{(2\sin\frac{t}{2})^2}\left(\frac{\alpha}{n^2} + \frac{2\alpha(1-\alpha)}{n^3\sin\frac{t}{2}}\right)\\ &< \frac{3\alpha}{2(2n\sin\frac{t}{2})^{\alpha+1}} \quad \left(\frac{\pi}{n}\leqslant t\leqslant\pi\right).\end{aligned} \tag{3}$$

Substituting (3) into (2) and noticing $\frac{1}{A_n^\alpha}\leqslant n^{-\alpha}$ (when $0<\alpha<1$) we get (1). □

Lemma 2. *Let* $0<\alpha<1$. *Define*

$$g_n^\alpha(t) := \int_0^t [K_n^\alpha(\theta) - K_{n-1}^\alpha(\theta)]\,d\theta.$$

Then

$$|g_n^\alpha(t)| \leqslant \begin{cases} 54t, & 0<t<\pi,\\ 40\alpha^{-1}n^{-\alpha-1}t^{-\alpha}, & \frac{\pi}{n}<t<\pi.\end{cases}$$

Proof. By the definition of $K_n^\alpha(t)$ we easily obtain

$$|K_n^\alpha(t) - K_{n-1}^\alpha(t)| \leqslant 1 + \frac{\alpha\Gamma(n+1)}{\Gamma(\alpha+n+1)}\sum_{k=1}^{n-1}\frac{(n-k)\Gamma(\alpha+k)}{\Gamma(k+1)} \leqslant 54.$$

This shows

$$|g_n^\alpha(t)| \leqslant \int_0^t |K_n^\alpha(\theta) - K_{n-1}^\alpha(\theta)|\,d\theta \leqslant 54\,t \quad (0<t<\pi).$$

We have

$$g_n^\alpha(t) = -\int_t^\pi [K_n^\alpha(\theta) - K_{n-1}^\alpha(\theta)]\,d\theta = -I_n(t) - \int_t^\pi \gamma_n^\alpha(\theta)\,d\theta, \tag{4}$$

where γ_n^α is defined in Lemma 1 and

$$I_n(t) = \frac{1}{A_n^\alpha}\int_t^\pi \frac{\cos[(n+\frac{\alpha}{2})\theta - \frac{\pi}{2}\alpha]}{(2\sin\frac{\theta}{2})^\alpha}\, d\theta.$$

Integrating by parts we derive

$$|I_n(t)| \leqslant \frac{1}{(n+\frac{\alpha}{2})A_n^\alpha}\left(\frac{2}{(2\sin\frac{t}{2})^\alpha} + \int_t^\pi \frac{\cos\frac{\theta}{2}}{(2\sin\frac{\theta}{2})^{\alpha+1}}\, d\theta\right) < \frac{1}{(n+\frac{\alpha}{2})A_n^\alpha}\frac{2+\frac{1}{\alpha}}{(2\sin\frac{t}{2})^\alpha}. \tag{5}$$

By (4) and (5), applying Lemma 1 we get

$$|g_n^\alpha(t)| \leqslant \frac{40}{\alpha n^{\alpha+1} t^\alpha} \quad \text{when } \frac{\pi}{n} < t < \pi. \qquad \square$$

Proof of the Theorem. We have

$$\begin{aligned} R_n^\alpha(f,x) &= \frac{1}{\pi}\sum_{j=n+1}^\infty \left|\int_0^\pi \varphi_x(t)\, dg_j^\alpha(t)\right| = \frac{1}{\pi}\sum_{j=n+1}^\infty \left|\int_0^\pi g_j^\alpha(t)\, d\varphi_x(t)\right| \\ &\leqslant \frac{1}{\pi}\sum_{j=n+1}^\infty \int_0^\pi |g_j^\alpha(t)|\, dV_0^t(\varphi_x) =: \frac{1}{\pi}(R_{n1}^\alpha + R_{n2}^\alpha), \end{aligned}$$

where

$$R_{n1}^\alpha = \sum_{k=n+1}^\infty \int_0^{\frac{\pi}{k}} |g_k^\alpha(t)|\, dV_0^t(\varphi_x), \quad R_{n2}^\alpha = \sum_{k=n+1}^\infty \int_{\frac{\pi}{k}}^\pi |g_k^\alpha(t)|\, dV_0^t(\varphi_x).$$

By Lemma 2 we have

$$R_{n1}^\alpha \leqslant 54\int_0^{\frac{\pi}{n+1}} \sum_{k=n+1}^\infty t\chi_{(0,\frac{\pi}{k})}(t)\, dV_0^t(\varphi_x) \leqslant 54\pi \int_0^{\frac{\pi}{n+1}} dV_0^t(\varphi_x) = 54\pi V_0^{\frac{\pi}{n+1}}(\varphi_x),$$

where $\chi_{(a,b)}$ denotes the characteristic function of (a,b). Also by Lemma 2 we have

$$\begin{aligned} R_{n2}^\alpha &\leqslant \frac{40}{\alpha}\sum_{k=n+1}^\infty \int_{\frac{\pi}{k}}^\pi \frac{1}{k^{\alpha+1}t^\alpha}\, dV_0^t(\varphi_x) = \frac{40}{\alpha}\sum_{k=n+1}^\infty \left(\int_{\frac{\pi}{k}}^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^\pi\right)\frac{1}{k^{\alpha+1}t^\alpha}\, dV_0^t(\varphi_x) \\ &= \frac{40}{\alpha}\left(\int_0^{\frac{\pi}{n+1}} \sum_{k>\frac{\pi}{t}} \frac{1}{k^{\alpha+1}t^\alpha}\, dV_0^t(\varphi_x) + \int_{\frac{\pi}{n+1}}^\pi \sum_{k=n+1}^\infty \frac{1}{k^{\alpha+1}t^\alpha}\, dV_0^t(\varphi_x)\right) \\ &\leqslant \frac{40}{\alpha^2}\left(V_0^{\frac{\pi}{n+1}}(\varphi_x) + \frac{1}{n^\alpha}\int_{\frac{\pi}{n+1}}^\pi \frac{1}{t^\alpha}\, dV_0^t(\varphi_x)\right) \\ &= \frac{40}{\alpha^2}\left(V_0^{\frac{\pi}{n+1}}(\varphi_x) + \frac{1}{n^\alpha}\sum_{k=1}^n \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} \frac{1}{t^\alpha}\, dV_0^t(\varphi_x)\right) \\ &\leqslant \frac{120}{\alpha^2 n^\alpha}\sum_{k=1}^n k^{\alpha-1} V_0^{\frac{\pi}{k}}(\varphi_x). \end{aligned}$$

Combining the estimates for R^{α}_{n1} and R^{α}_{n2} we obtain

$$R^{\alpha}_{n}(f,x) \leqslant \frac{100}{\alpha^2 n^{\alpha}} \sum_{k=1}^{n} k^{\alpha-1} V_0^{\frac{\pi}{k}}(\varphi_x) \quad (n \geqslant 2).$$

On the other hand, the function

$$f^*(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} = \begin{cases} \frac{\pi - x}{2} & \text{when } 0 < x < 2\pi, \\ 0 & \text{when } x = 0 \end{cases}$$

gives the second conclusion of the Theorem at point $x = \frac{\pi}{2}$. □

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